

Genus Two Virasoro Correlation Functions for Vertex Operator Algebras

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Abstract

We consider all genus two correlation functions for the Virasoro vacuum descendants of a vertex operator algebra. These are described in terms of explicit generating functions that can be combinatorially expressed in terms of a sequence of globally defined differential operators on which the genus two Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ has a natural action.

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1 Introduction

A Vertex Operator Algebra (VOA) (e.g. [FLM], [K], [LL], [MT1]) is an algebraic system related to Conformal Field Theory (CFT) in physics e.g. [DMS]. An essential ingredient of a VOA or CFT is the existence of a conformal Virasoro vector whose vertex operator modes generate the Virasoro algebra of central charge c . The connection between VOAs, the Virasoro vector and genus one Riemann surfaces was established in the foundational work of Zhu [Z] giving a rigorous basis for many ideas in CFT. Zhu established a general recursion formula relating any genus one n -point correlation function to linear combination of $(n-1)$ -point correlation functions. When applied to genus one Virasoro correlation functions, Zhu recursion implies the genus one Ward identities. For a suitable VOA (e.g. if V is C_2 cofinite [Z]) the Ward identities imply the partition function for V and its ordinary modules satisfy a genus one modular ordinary differential equation. This paper lays some of the ground work for the development of a new theory of genus two $\mathrm{Sp}(4, \mathbb{Z})$ Siegel modular partial differential equation for the genus two partition function of a suitable VOA V and its ordinary modules. This is illustrated in a sequel paper where we describe a new $\mathrm{Sp}(4, \mathbb{Z})$ modular partial differential equation for the partition functions (for V and its ordinary modules) for the $(2, 5)$ minimal Virasoro model of central charge $c = -\frac{22}{5}$ [GT2].

Correlation functions for VOAs on a genus two Riemann surface have been defined, and in some cases calculated based on an explicit sewing procedure for sewing two tori together [MT2, MT3]. We recently described a general Zhu recursion formula for genus two n -point correlation functions which gives rise to new genus two Ward identities when applied to Virasoro vector n -point functions [GT1]. The purpose of this paper is to describe all genus two correlation functions for Virasoro descendants of the vacuum vector in terms of explicit generating functions. These generating functions, which satisfy the genus two Ward identity, are shown to be combinatorially expressed in terms of a sequence of globally defined differential operators on which the Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ has an natural action.

We begin in Section 2 with a very brief review of VOAs on a genus two Riemann surface. We review a sewing scheme for constructing a genus two surface from two punctured tori and how to formally define the genus two partition and n -point correlation functions in terms of genus one VOA data [MT2, MT3].

In Section 3 contains the main results of this paper. We first show that genus two n -point correlation functions for n Virasoro vectors are generating functions for the correlation functions for all Virasoro vacuum descendants in a similar fashion to the genus zero and one cases [HT1]. These generating functions satisfy genus two Ward identities (derived from genus two Zhu recursion) which involves genus two *generalised Weierstrass functions* related to a global $(2,1)$ -bidifferential $\Psi(x, y)$ holomorphic for $x \neq y$ [GT1]. We also describe some analytic differential equations, which also involve $\Psi(x, y)$, for the genus two bidifferential $\omega(x, y)$, normalised holomorphic 1-differentials $\nu_1(x), \nu_2(x)$ and the projective connection $s(x)$. Using these differential equations, we demonstrate in Theorem 3.11 how to express each generating function in a symmetric way as a sum of given weights of appropriate graphs. In particular, the Virasoro

vector n -point function is determined by the action of a specific symmetric differential operator \mathcal{O}_n on the normalised partition function $Z_V^{(2)}/(Z_M^{(2)})^c$ (cf. (3.15)) where $Z_V^{(2)}$ denotes the genus two partition function for V , $Z_M^{(2)}$ denotes the genus two partition function for the Heisenberg VOA M and c is the central charge.

Lastly in Section 4 we consider general analytic and modular transformation properties of the differential operator \mathcal{O}_n in any coordinate system on an arbitrary genus two Riemann surface. In particular, in Theorem 4.6, we show how \mathcal{O}_n transforms under the Siegel modular group $\mathrm{Sp}(4, \mathbb{Z})$ having established that the global $(2,1)$ -bidifferential $\Psi(x, y)$ is $\mathrm{Sp}(4, \mathbb{Z})$ invariant in Theorem 4.2.

2 Vertex operator algebras on a genus two Riemann surface

2.1 Genus two Riemann surfaces

We briefly review some concepts in genus two Riemann surface theory e.g. [FK, F, Mu1]. Let $\mathcal{S}^{(2)}$ be a compact genus two Riemann surface with canonical homology basis α^i, β^i for $i = 1, 2$. There exists a unique holomorphic symmetric bidifferential $(1,1)$ -form $\omega(x, y)$, the *normalised bidifferential of the second kind*, where for $x \neq y \in \mathcal{S}^{(2)}$

$$\omega(x, y) = \frac{dxdy}{(x-y)^2} + \frac{1}{6}s(x) - \frac{(x-y)}{12}\partial_x s(x) + O((x-y)^2), \quad (2.1)$$

$$\oint_{\alpha^i} \omega(x, \cdot) = 0, \quad i = 1, 2.$$

$s(x)$ is the *projective connection* transforming under an analytic map $x \rightarrow \phi(x)$ as

$$s(x) = s(\phi(x)) + \{\phi(x), x\}dx^2, \quad (2.2)$$

where $\{\phi(x), x\} = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2} \left(\frac{\phi''(x)}{\phi'(x)} \right)^2$ is the Schwarzian derivative. Furthermore

$$\nu_i(x) = \oint_{\beta^i} \omega(x, \cdot), \quad \Omega_{ij} = \frac{1}{2\pi i} \oint_{\beta^i} \nu_j, \quad i, j = 1, 2.$$

for *holomorphic differentials* $\nu_i(x)$ normalised by $\oint_{\beta^i} \nu_j = 2\pi i \delta_{ij}$ and *period matrix* $\Omega \in \mathbb{H}_2$, the genus two Siegel upper half plane i.e. $\Omega = \Omega^T$ and $\Im(\Omega) > 0$.

Consider the genus two Riemann surface $\mathcal{S}^{(2)}$ constructed by sewing two genus one tori $\mathcal{S}_a = \mathbb{C}/\Lambda_a$, for lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ with modular parameter $\tau_a \in \mathbb{H}_1$ for $a = 1, 2$ [MT2]. Let $z_a \in \mathcal{S}_a$, $\epsilon \in \mathbb{C}$ and define punctured tori

$$\widehat{\mathcal{S}}_1 = \mathcal{S}_1 \setminus \{z_1, |z_1| \leq |\epsilon|/r_2\}, \quad \widehat{\mathcal{S}}_2 = \mathcal{S}_2 \setminus \{z_2, |z_2| \leq |\epsilon|/r_1\},$$

where $|\epsilon| \leq r_1 r_2$. We identify the annular regions $\{z_1, |\epsilon|/r_2 \leq |z_1| \leq r_1\}$ and $\{z_2, |\epsilon|/r_1 \leq |z_2| \leq r_2\}$ via the sewing relation $z_1 z_2 = \epsilon$. Then $\mathcal{S}^{(2)}$ is parameterized by the sewing domain

$$\mathcal{D}_{\text{sew}} = \left\{ (\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} : |\epsilon| < \frac{1}{4} D(q_1) D(q_2) \right\}, \quad (2.3)$$

where $q_a = e^{2\pi i \tau_a}$ and $D(q_a) = \min_{\lambda_a \in \Lambda_a, \lambda_a \neq 0} |\lambda_a|$. We may then obtain explicit expressions for $\omega(x, y)$, $\nu_i(x)$ and Ω_{ij} on $\mathcal{S}^{(2)}$ for $x, y \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$ described in [MT2].

2.2 Vertex operator algebras on a torus

We review aspects of vertex operator algebras (e.g. [FLM, K, LL, MT1]). A Vertex Operator Algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a \mathbb{Z} -graded complex vector space $V = \bigoplus_{n \in \mathbb{Z}} V(n)$ where $\dim V(n) < \infty$ for each $n \in \mathbb{Z}$, a linear map $Y : V \rightarrow \text{End}(V)[[z, z^{-1}]]$ for a formal parameter z and pair of distinguished vectors: the vacuum $\mathbf{1} \in V_{(0)}$ and the conformal vector $\omega \in V_{(2)}$. For each $v \in V$, the image under the map Y is the *vertex operator*

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1},$$

with *modes* $v(n) \in \text{End}(V)$, where $Y(v, z)\mathbf{1} = v + O(z)$. Vertex operators satisfy *locality* i.e. for all $u, v \in V$ there exists an integer $k \geq 0$ such that

$$(z_1 - z_2)^k [Y(u, z_1), Y(v, z_2)] = 0.$$

The vertex operator of the conformal vector ω is $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ where the modes $L(n)$ satisfy the Virasoro algebra with *central charge* c

$$[L(m), L(n)] = (m - n)L(m + n) + c \frac{m^3 - m}{12} \delta_{m, -n} \text{Id}_V.$$

Furthermore, $L(0)v = kv$ for *conformal weight* $\text{wt}(v) = k$ for all $v \in V_{(k)}$ and $Y(L(-1)u, z) = \partial_z Y(u, z)$.

In order to describe VOAs on a torus, Zhu [Z] introduced an isomorphic VOA $(V, Y[\ , \], \mathbf{1}, \tilde{\omega})$ with “square bracket” vertex operators

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(e^{L(0)} v, e^z - 1),$$

and conformal vector $\tilde{\omega} = \omega - \frac{c}{24} \mathbf{1}$ with Virasoro modes $L[n]$.

Define the *genus one partition function* by the formal trace $Z_V^{(1)}(\tau) = \text{Tr}_V (q^{L(0)-c/24})$, with $q = e^{2\pi i \tau}$, the *genus one correlation one point function* by the formal trace

$$Z_V^{(1)}(v; \tau) = \text{Tr}_V (o(v) q^{L(0)-c/24}), \quad v \in V,$$

where $o(v) = v(k-1)$ for $v \in V_{(k)}$ and the *genus one n -point correlation function* for $v_1, \dots, v_n \in V$ inserted at $z_1, \dots, z_n \in \mathbb{C}/(2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z}))$ by

$$Z_V^{(1)}(v_1, z_1; \dots; v_n, z_n; \tau) = Z_V^{(1)}(Y[v_1, z_1] \dots Y[v_n, z_n] \mathbf{1}; \tau).$$

Zhu describes a general recursion formula for expressing any genus one n -point correlation function as a linear combination of $(n-1)$ -point functions with universal coefficients given by explicit elliptic functions [Z]. All of the above definitions can be naturally extended to define $Z_W^{(1)}(\dots)$ for an ordinary graded V -module W where the trace is taken over W .

2.3 VOAs on a genus two Riemann surface

We define the genus two partition function and n -point correlation function for a VOA based on the sewing scheme for $\mathcal{S}^{(2)}$ constructed from two tori \mathcal{S}_1 and \mathcal{S}_2 [MT3, GT1]. The *genus two partition function* for V of strong CFT-type is defined by

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{u \in V} Z_V^{(1)}(u; \tau_1) Z_V^{(1)}(\bar{u}; \tau_2), \quad (2.4)$$

where the formal sum is taken over any V -basis and \bar{u} is the dual of u with respect to an invariant invertible bilinear form $\langle \cdot, \cdot \rangle$ associated with the Mobius map $z \rightarrow \epsilon/z$ (see [GT1] for more details).

The *genus two n -point correlation function* for $a_1, \dots, a_L \in V$ and $b_1, \dots, b_R \in V$ formally inserted at $x_1, \dots, x_L \in \widehat{\mathcal{S}}_1$ and $y_1, \dots, y_R \in \widehat{\mathcal{S}}_2$, respectively, is defined by

$$\begin{aligned} & Z_V^{(2)}(a_1, x_1; \dots; a_L, x_L | b_1, y_1; \dots; b_R, y_R; \tau_1, \tau_2, \epsilon) \\ &= \sum_{u \in V} Z_V^{(1)}(Y[a_1, x_1] \dots Y[a_L, x_L] u; \tau_1) Z_V^{(1)}(Y[b_R, y_R] \dots Y[b_1, y_1] \bar{u}; \tau_2). \end{aligned} \quad (2.5)$$

Convergent expressions have been found for such correlation functions for particular VOAs such as the Heisenberg VOA and lattice VOAs [MT3]. A formal Zhu recursion formula for a genus two n -point function in terms of $(n-1)$ -point functions is described in [GT1] where the coefficients are formal series, called *generalised Weierstrass functions*, which depend on the conformal weight of the recursion vector but are otherwise universal. For conformal weight 1 or 2, these series are holomorphic on appropriate domains [GT1]. The above definition can be naturally extended to define $Z_{W_1, W_2}^{(2)}(\dots)$ for a pair of ordinary graded V -modules W_1, W_2 where the left or right hand trace is taken over W_1 or W_2 respectively.

3 Genus Two Virasoro Correlation Functions

3.1 The Virasoro generating function

Consider the genus two Virasoro n -point correlation function for $z_1, \dots, z_n \in \widehat{\mathcal{S}}_1$

$$Z_V^{(2)}(\tilde{\omega}, z_1; \dots; \tilde{\omega}, z_n) = \sum_{u \in V} Z_V^{(1)}(Y[\tilde{\omega}, z_1] \dots Y[\tilde{\omega}, z_n] u; \tau_1) Z_V^{(1)}(\bar{u}; \tau_2), \quad (3.1)$$

(where we suppress the dependence on τ_1, τ_2, ϵ). We define the formal differential

$$G_n(\mathbf{z}) = G_n(z_1, \dots, z_n) = Z_V^{(2)}(\tilde{\omega}, z_1; \dots; \tilde{\omega}, z_n) d\mathbf{z}^2, \quad (3.2)$$

where $d\mathbf{z}^2 = dz_1^2 \dots dz_n^2$. $G_n(\mathbf{z})$ is independent of whether we formally insert $\tilde{\omega}$ at $z_i \in \hat{\mathcal{S}}_1$ or at $\epsilon/z_i \in \hat{\mathcal{S}}_2$ (Proposition 6 of [MT3]). Similarly to [HT1] we find

Proposition 3.1. *$G_m(\mathbf{z})$ is symmetric in z_i and is a generating function for all genus two n -point correlation functions for Virasoro vacuum descendants.*

Proof. $G_m(\mathbf{z})$ is symmetric in z_1, \dots, z_m by locality. Consider the genus two n -point function for n Virasoro vacuum descendants $v_i = L[-k_{i1}] \dots L[-k_{im_i}] \mathbf{1}$ inserted at $z_i \in \hat{\mathcal{S}}_1$ for $i = 1, \dots, n$ and $k_{ij} \geq 2$ given by

$$Z_V^{(2)}(v_1, z_1; \dots; v_n, z_n) = \sum_{u \in V} Z_V^{(1)}(Y[v_1, z_1] \dots Y[v_n, z_n] u; \tau_1) Z_V^{(1)}(\bar{u}; \tau_2).$$

$Z_V^{(1)}(Y[v_1, z_1] \dots Y[v_n, z_n] u; \tau_1)$ is the coefficient of $\prod_{i=1}^n \prod_{j=1}^{m_i} (x_{ij})^{k_{ij}-2}$ in

$$Z_V^{(1)}(Y[Y[\tilde{\omega}, x_1] \dots Y[\tilde{\omega}, x_{1m_1}] \mathbf{1}, z_1] \dots Y[Y[\tilde{\omega}, x_{n1}] \dots Y[\tilde{\omega}, x_{nm_n}] \mathbf{1}, z_n] u; \tau_1).$$

Using associativity and lower truncation (e.g. [K, LL, MT1]) we find for $N \gg 0$ that

$$\begin{aligned} & \prod_{i=1}^n \prod_{j=1}^{m_i} (x_{ij} + z_i)^N Y[Y[\tilde{\omega}, x_{11}] \dots Y[\tilde{\omega}, x_{1m_1}] \mathbf{1}, z_1] \dots Y[Y[\tilde{\omega}, x_{n1}] \dots Y[\tilde{\omega}, x_{nm_n}] \mathbf{1}, z_n] u \\ &= \prod_{i=1}^n \prod_{j=1}^{m_i} (x_{ij} + z_i)^N Y[\tilde{\omega}, z_1 + x_{11}] \dots Y[\tilde{\omega}, z_1 + x_{1m_1}] \dots Y[\tilde{\omega}, z_n + x_{n1}] \dots Y[\tilde{\omega}, z_n + x_{nm_n}] u. \end{aligned}$$

Thus the genus two n -point function for v_1, \dots, v_n is the coefficient of $\prod_{i=1}^n \prod_{j=1}^{m_i} (x_{ij})^{k_{ij}-2}$ of the formal expansion of $Z_V^{(2)}(\tilde{\omega}, z_1 + x_{11}; \dots; \tilde{\omega}, z_n + x_{nm_n})$ for $M = \sum_{i=1}^n m_i$. \square

3.2 A genus two Ward identity

Define a genus two modular derivative operator

$$\nabla_x = \sum_{1 \leq a \leq b \leq 2} \nu_a(x) \nu_b(x) \frac{\partial}{\partial \Omega_{ab}}, \quad (3.3)$$

for period matrix Ω_{ab} , and normalised holomorphic 1-differentials ν_a . There exists an injective but non-surjective holomorphic map F^Ω from the sewing domain \mathcal{D}_{sew} into the Siegel upper half plane [GT1]

$$\begin{aligned} F^\Omega : \mathcal{D}_{\text{sew}} &\rightarrow \mathbb{H}_2, \\ (\tau_1, \tau_2, \epsilon) &\mapsto \Omega(\tau_1, \tau_2, \epsilon), \end{aligned} \quad (3.4)$$

Below we will also denote by ∇_x the action of $(F^\Omega)^{-1} \circ \nabla_x \circ F^\Omega$ on \mathcal{D}_{sew} .

In Section 5 of [GT1] we describe a genus two Ward identity for the genus two Virasoro n -point correlation function. This is expressed in terms of generalised Weierstrass functions ${}^2\mathcal{P}_k(x, y)$ for $k \geq 1$ defined as follows. Let $\boldsymbol{\nu}(x) = [\nu_1(x), \nu_2(x)]$ denote a row vector of holomorphic 1-differentials and define

$$\Psi(x, y) = {}^2\mathcal{P}_1(x, y)dx^2(dy)^{-1} = - \frac{\omega(x, y) \left| \frac{\boldsymbol{\nu}(x)}{\boldsymbol{\nu}(y)} \right| + \left| \frac{\boldsymbol{\nu}(y)}{\nabla_x \boldsymbol{\nu}(y)} \right|}{\left| \frac{\boldsymbol{\nu}(y)}{\partial_y \boldsymbol{\nu}(y)} \right| dy}. \quad (3.5)$$

Proposition 3.2 ([GT1]). $\Psi(x, y)$ is a holomorphic $(2, -1)$ -bidifferential for $x \neq y$ where, for any local coordinates x, y

$$\Psi(x, y) = \left(\frac{1}{x - y} + \text{regular terms} \right) dx^2(dy)^{-1}.$$

We also define generalised Weierstrass functions for $k \geq 1$ by

$${}^2\mathcal{P}_k(x, y) = \frac{1}{(k-1)!} \partial_y^{k-1} ({}^2\mathcal{P}_1(x, y)) = \frac{1}{(x-y)^k} + \text{regular terms}, \quad (3.6)$$

which is holomorphic for $x \neq y$.

Proposition 3.3 ([GT1]). $G_n(\mathbf{z})$ obeys the formal Ward identity for $z_1, \dots, z_n \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$ and $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}_{\text{sew}}$

$$\begin{aligned} G_n(\mathbf{z}) = & \left(\nabla_{z_1} + dz_1^2 \sum_{k=2}^n ({}^2\mathcal{P}_1(z_1, z_k) \partial_{z_k} + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k)) \right) G_{n-1}(z_2, \dots, z_n) \\ & + \frac{c}{2} \sum_{k=2}^n {}^2\mathcal{P}_4(z_1, z_k) G_{n-2}(z_2, \dots, \widehat{z}_k, \dots, z_n) dz_1^2 dz_k^2, \end{aligned} \quad (3.7)$$

where \widehat{z}_k denotes the omission of the given term.

We note that the above expression for $G_n(\mathbf{z})$ is not manifestly symmetric in its arguments.

3.3 Some analytic differential equations

For the genus two bidifferential $\omega(x, y)$, normalised holomorphic 1-differentials $\nu_a(x)$ for $a = 1, 2$ and the projective connection $s(x)$ we find from Section 6 of [GT1] that

Proposition 3.4. $\omega(x, y)$, $\nu_a(x)$ for $a=1,2$ and $s(x)$ satisfy the following analytic differential equations

$$\left(\nabla_x + dx^2 \sum_{r=1}^2 ({}^2\mathcal{P}_1(x, y_r) \partial_{y_r} + {}^2\mathcal{P}_2(x, y_r)) \right) \omega(y_1, y_2) = \omega(x, y_1) \omega(x, y_2), \quad (3.8)$$

$$\left(\nabla_x + dx^2 ({}^2\mathcal{P}_1(x, y) \partial_y + {}^2\mathcal{P}_2(x, y)) \right) \nu_a(y) = \omega(x, y) \nu_a(x), \quad (3.9)$$

$$\left(\nabla_x + dx^2 ({}^2\mathcal{P}_1(x, y) \partial_y + 2 {}^2\mathcal{P}_2(x, y)) \right) s(y) + 6 {}^2\mathcal{P}_4(x, y) dx^2 dy^2 = 6 \omega(x, y)^2, \quad (3.10)$$

for all $x, y_1, y_2 \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$ and $\Omega \in F^\Omega(\mathcal{D}_{\text{sew}})$.

We may generalise (3.8) and (3.9) in the following coordinate independent way:

Corollary 3.5. $\omega(x, y)$ and $\nu_a(x)$ for $a=1,2$ satisfy the following coordinate independent analytic differential equations for all $\Omega \in \mathbb{H}_2$

$$\nabla_x \omega(y_1, y_2) + \sum_{r=1}^2 \partial_{y_r} (\Psi(x, y_r) \omega(y_1, y_2)) dy_r = \omega(x, y_1) \omega(x, y_2), \quad (3.11)$$

$$\nabla_x \nu_a(y) + \partial_y (\Psi(x, y) \nu_a(y)) dy = \omega(x, y) \nu_a(x). \quad (3.12)$$

Proof. $\Psi(x, y_1) \omega(y_1, y_2)$ is a global $(2, 0, 1)$ -form in (x, y_1, y_2) (with a similar statement for $\Psi(x, y_2) \omega(y_1, y_2)$). Thus

$$d_{y_r} (\Psi(x, y_r) \omega(y_1, y_2)) = \partial_{y_r} (\Psi(x, y_r) \omega(y_1, y_2)) dy_r,$$

is a global $(2, 1, 1)$ -form. Hence (3.8) can be expressed by (3.11) in a coordinate independent way for all $\Omega \in F^\Omega(\mathcal{D}_{\text{sew}})$. But all parts of (3.11) are holomorphic for all $\Omega \in \mathbb{H}_2$ and hence the identity can be analytically extended from $F^\Omega(\mathcal{D}_{\text{sew}})$ to \mathbb{H}_2 . (3.12) follows from (3.11) by integrating y_2 along the β^a homology cycle. \square

We may also generalise (3.10) in the following way:

Corollary 3.6. $s(y)$, for a given choice of local coordinate y , satisfies the following analytic differential equation for all $\Omega \in \mathbb{H}_2$

$$\left(\nabla_x + dy (\Psi(x, y) \partial_y + 2 \partial_y \Psi(x, y)) \right) s(y) + dy^3 \partial_y^3 \Psi(x, y) = 6 \omega(x, y)^2. \quad (3.13)$$

Proof. We let $y_2 = y$ and $y_1 = y + \varepsilon$ and note from (2.1) that

$$\begin{aligned} \omega(y_1, y_2) &= \frac{dy^2}{\varepsilon^2} + \frac{1}{6} s(y) + O(\varepsilon), \\ \partial_{y_1} \omega(y_1, y_2) &= -\frac{2dy^2}{\varepsilon^3} + \frac{1}{12} \partial_y s(y) + O(\varepsilon), \\ \partial_{y_2} \omega(y_1, y_2) &= \frac{2dy^2}{\varepsilon^3} + \frac{1}{12} \partial_y s(y) + O(\varepsilon), \\ \Psi(x, y_1) &= \Psi(x, y) + \varepsilon \partial_y \Psi(x, y) + \frac{\varepsilon^2}{2} \partial_y^2 \Psi(x, y) + \frac{\varepsilon^3}{6} \partial_y^3 \Psi(x, y) + O(\varepsilon^4), \\ \partial_y \Psi(x, y_1) &= \partial_y \Psi(x, y) + \varepsilon \partial_y^2 \Psi(x, y) + \frac{\varepsilon^2}{2} \partial_y^3 \Psi(x, y) + O(\varepsilon^3). \end{aligned}$$

The result follows by substituting the above into (3.11) and taking the $\varepsilon \rightarrow 0$ limit. \square

We finally note that the genus two partition function $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ for the Heisenberg VOA M obeys [GT1]

Proposition 3.7. $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ is holomorphic for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}_{\text{sew}}$ and satisfies

$$\nabla_x Z_M^{(2)} = \frac{1}{12} s(x) Z_M^{(2)}, \quad (3.14)$$

for $x \in \widehat{\mathcal{S}}_1 \cup \widehat{\mathcal{S}}_2$.

Remark 3.8. $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ can be considered as a holomorphic function on $F^\Omega(\mathcal{D}_{\text{sew}})$ but **cannot** be analytically continued to the full Siegel upper half plane \mathbb{H}_2 (cf. Theorem 7.2, [GT1]). In physics, this follows from the conformal anomaly (e.g. [FS]) which for the Heisenberg VOA is believed to be related to the non-existence of a global section of certain determinant line bundles on the genus two Riemann surface [Mu2].

(3.8)-(3.14) are the genus two analogues of differential equations for elliptic and modular functions described in [HT1]. Thus (3.14) corresponds to

$$q \frac{\partial}{\partial q} \left(\frac{1}{\eta(q)} \right) = \frac{1}{2} E_2(q) \left(\frac{1}{\eta(q)} \right),$$

for the weight 2 quasi-modular Eisenstein series $E_2(q) = -\frac{1}{12} + 2 \sum_{m,n \geq 1} n q^{mn}$.

3.4 The main theorem

We show below in Theorem 3.11 how to express $G_n(\mathbf{z})$ in a manifestly symmetric fashion as a sum of weights of appropriate graphs. The graph configurations are precisely those exploited in [HT1] to describe genus one Virasoro n -point functions and many of the arguments below mirror the genus one case. However, the graph weights are differently defined in the genus two case and the technicalities are more involved. Furthermore, the genus two graph weights for $G_n(\mathbf{z})$ are described in terms of a linear differential operator $\mathcal{O}_n(\mathbf{z})$ which is symmetric in its arguments and possesses fundamental properties under analytic and genus two $\text{Sp}(4, \mathbb{Z})$ modular transformations.

Define, for central charge c , a formal normalised partition function

$$\Theta_V(\tau_1, \tau_2, \epsilon) := Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^{-c} Z_V^{(2)}(\tau_1, \tau_2, \epsilon), \quad (3.15)$$

where $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ is the genus two partition function for the Heisenberg VOA (which is holomorphic on \mathcal{D}_{sew}). Following Remark 3.8 and [FS] we conjecture:

Conjecture 3.9. Θ_V is holomorphic on \mathbb{H}_2 for a C_2 -cofinite VOA V . If V is also rational then Θ_V is a component of a vector valued Siegel modular form of weight $c/2$.

For example, for a lattice VOA V_L for an even lattice L of rank c we find $\Theta_V = \Theta_L(\Omega)$, the genus two Siegel lattice theta function [MT3], a Siegel modular form of weight $c/2$ and level $|L^*/L|$ where L^* is the dual lattice. Thus, conjecturally, it is the normalised partition function Θ_V on which the full genus two $\text{Sp}(4, \mathbb{Z})$ modular group naturally acts.¹ One of the main purposes of this paper is to develop global differential operators

¹We also note that Θ_V has canonical properties in the one torus degeneration limite [HT2].

$\mathcal{O}_n(\mathbf{z})$ on which $\text{Sp}(4, \mathbb{Z})$ acts. In the sequel [GT2] we show how these operators give rise to a $\text{Sp}(4, \mathbb{Z})$ modular differential equation for the partition function for the $(2, 5)$ minimal Virasoro model of central charge $-22/5$.

We define the linear differential operator $\mathcal{O}_n(\mathbf{z})$ (which in general acts on differentiable functions of Ω) by

$$\mathcal{O}_n(\mathbf{z})\Theta_V := Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^{-c} G_n(\mathbf{z}). \quad (3.16)$$

For $n = 1$ we find $G_1(z_1) = \nabla_{z_1} Z_V^{(2)}(\tau_1, \tau_2, \epsilon)$ so that, using (3.14), we find²

$$\mathcal{O}_1(z_1) = \nabla_{z_1} + \frac{c}{12} s(z_1). \quad (3.17)$$

In order to describe the $n = 2$ case, we also define the differential operator

$$\mathcal{D}_{z_1, z_2} = \nabla_{z_1} + dz_1^2 ({}^2\mathcal{P}_1(z_1, z_2) \partial_{z_2} + 2 {}^2\mathcal{P}_2(z_1, z_2)). \quad (3.18)$$

Then for $n = 2$, the Ward identity (3.7) implies

$$\begin{aligned} \mathcal{O}_2(z_1, z_2)\Theta_V &= \left(Z_M^{(2)}\right)^{-c} \mathcal{D}_{z_1, z_2} \nabla_{z_2} Z_V^{(2)} + \frac{c}{2} {}^2\mathcal{P}_4(z_1, z_2) \Theta_V dz_1^2 dz_2^2 \\ &= \mathcal{D}_{z_1, z_2} \left(\nabla_{z_2} \Theta_V + \frac{c}{12} s(z_2) \Theta_V \right) + \frac{c}{12} s(z_1) \nabla_{z_2} \Theta_V \\ &\quad + \frac{c^2}{144} s(z_1) s(z_2) \Theta_V + \frac{c}{2} {}^2\mathcal{P}_4(z_1, z_2) dz_1^2 dz_2^2 \Theta_V. \end{aligned}$$

From (3.9) we note that

$$\mathcal{D}_{z_1, z_2} \nu_a(z_2) \nu_b(z_2) = \omega(z_1, z_2) (\nu_a(z_1) \nu_b(z_2) + \nu_a(z_2) \nu_b(z_1)). \quad (3.19)$$

(3.19) together with (3.10) imply that

$$\begin{aligned} \mathcal{O}_2(z_1, z_2) &= \sum_{1 \leq a \leq b \leq 2} \sum_{1 \leq c \leq d \leq 2} \nu_a(z_1) \nu_b(z_1) \nu_c(z_2) \nu_d(z_2) \frac{\partial^2}{\partial \Omega_{ab} \partial \Omega_{cd}} \\ &\quad + \frac{c}{12} s(z_1) \nabla_{z_2} + \frac{c}{12} s(z_2) \nabla_{z_1} + \frac{c^2}{144} s(z_1) s(z_2) \\ &\quad + 2 \omega(z_1, z_2) \sum_{1 \leq a \leq b \leq 2} \nu_a(z_1) \nu_b(z_2) \frac{\partial}{\partial \Omega_{ab}} + \frac{c}{2} \omega(z_1, z_2)^2. \end{aligned} \quad (3.20)$$

This expression is clearly symmetric in z_1, z_2 in accordance with Proposition 3.1. Furthermore, each term in (3.20) is now written in coordinate independent way.

Similarly to Section 3 of [HT1] we now develop a graphical/combinatorial approach for computing $\mathcal{O}_n(\mathbf{z})$ and hence G_n for all n . We define an *order n Virasoro graph* to be a directed graph \mathcal{G}^n with n vertices labelled by z_1, \dots, z_n . Each z_i -vertex has degree $\deg(z_i) = 0, 1$ or 2 . The degree 1 vertices can have either unit indegree or outdegree whereas the degree 2 vertices have both unit indegree and outdegree. Thus, the connected subgraphs of \mathcal{G}^n consist of r -cycles, with $r \geq 1$ degree 2 vertices, and chains with two degree 1 end-vertices with all vertices of degree 2. We regard a single degree 0 vertex as being a degenerate chain.

² The operator (3.17) is like a higher genus Serre derivative as discussed further in [GT1].

Remark 3.10. *The set of non-isomorphic order n Virasoro graphs is in one to one correspondence with the set of partial permutations of the label set $\{1, \dots, n\}$. This is described in further detail in [HT1].*

We define a genus two weight $W(\mathcal{G}^n)$ on \mathcal{G}^n as follows. For each directed edge \mathcal{E}_{ij} we define an edge weight

$$W(\mathcal{E}_{ij}) = W(z_i \longrightarrow z_j) = \begin{cases} \frac{1}{6}s(z_i) & \text{for } i = j, \\ \omega(z_i, z_j) & \text{for } i \neq j. \end{cases} \quad (3.21)$$

Let $\mathcal{C}_{k\ell}$ denote a chain in \mathcal{G}^n with end-vertices z_k and z_ℓ

$$z_k \circ \xrightarrow{z_m} \circ \dots \circ \xrightarrow{z_n} \circ z_\ell$$

and assign a chain weight (including the degenerate chain)

$$W(\mathcal{C}_{k\ell}) = W(z_k \circ \xrightarrow{\quad} \circ \dots \circ \xrightarrow{\quad} \circ z_\ell) = A(z_k, z_\ell), \quad (3.22)$$

where $A(z_k, z_\ell) = \sum_{1 \leq a \leq b \leq 2} \nu_a(z_k) \nu_b(z_\ell) \alpha_{ab}$ for free parameters $\alpha_{ab} = \alpha_{ba}$. Let K be the number of cycles and define a weight for \mathcal{G}^n by

$$W(\mathcal{G}^n) = \left(\frac{c}{2}\right)^K \prod_{\mathcal{E}_{ij}} W(\mathcal{E}_{ij}) \prod_{\mathcal{C}_{k\ell}} W(\mathcal{C}_{k\ell}), \quad (3.23)$$

where the first product ranges over all the edges and the second product ranges over all the chains of \mathcal{G}^n . Thus the weight depends on c , $\omega(z_i, z_j)$, $s(z_i)$, $\nu_a(z_i)$ and α_{ab} . We also note that W is multiplicative on the disconnected components of \mathcal{G}^n .

Lastly, define a linear map \mathcal{L}_α from $\mathbb{C}[\alpha_{ab}]$, the vector space of complex coefficient polynomials in α_{ab} , to the complex vector space spanned by $\frac{\partial}{\partial \Omega_{ab}}$ derivatives with

$$\mathcal{L}_\alpha(\alpha_{a_1 b_1} \dots \alpha_{a_M b_M}) = \frac{\partial^M}{\partial \Omega_{a_1 b_1} \dots \partial \Omega_{a_M b_M}}. \quad (3.24)$$

Let p_{KM}^n be the number of inequivalent order n Virasoro graphs containing K cycles and M chains. In [HT1] the following graph generating function is established

$$p^n(\alpha, \beta) = \sum_{K \geq 0, M \geq 0} p_{KM}^n \alpha^K \beta^M = (-1)^n n! \sum_{i=0}^n \frac{(-\alpha)^i}{i!} \binom{-\beta - i}{n - i}, \quad (3.25)$$

for chain and cycle counting parameters α and β respectively. Thus for $n = 1$ we find $p^1(\alpha, \beta) = \alpha + \beta$ corresponding to two inequivalent graphs with weights

$$W(z_1 \circ) = A(z_1, z_1), \quad W\left(z_1 \circ \begin{array}{c} \circ \\ \text{---} \end{array}\right) = \frac{c}{2} \frac{s(z_1)}{6},$$

whose weight sum under the action of \mathcal{L}_α is $\mathcal{O}_1(z_1)$ using (3.17).

For $n = 2$ we have $p^2(\alpha, \beta) = \alpha^2 + 2\alpha\beta + \beta^2 + \beta + 2\alpha$ for 7 graphs with weights:

$$\begin{aligned}
W\left(z_1 \circ \circ z_2\right) &= A(z_1, z_1)A(z_2, z_2), \\
W\left(z_1 \circ \text{loop} \circ z_2\right) &= \frac{c}{2} \frac{s(z_1)}{6} A(z_2, z_2), \quad W\left(z_1 \circ \text{loop} z_2\right) = \frac{c}{2} \frac{s(z_2)}{6} A(z_1, z_1), \\
W\left(z_1 \text{loop} \text{loop} z_2\right) &= \left(\frac{c}{2}\right)^2 \frac{s(z_1)}{6} \frac{s(z_2)}{6}, \quad W\left(z_1 \circ \text{loop} \circ z_2\right) = \frac{c}{2} \omega(z_1, z_2)^2, \\
W\left(z_1 \circ \leftarrow \circ z_2\right) &= W\left(z_1 \circ \rightarrow \circ z_2\right) = A(z_1, z_2),
\end{aligned}$$

whose weight sum under the action of \mathcal{L}_α using (3.20) is

$$\sum_{\mathcal{G}^2} \mathcal{L}_\alpha (W(\mathcal{G}^2)) = \mathcal{O}_2(z_1, z_2).$$

These examples illustrate the general result:

Theorem 3.11. *The order n genus two Virasoro generating function is given by*

$$G_n(\mathbf{z}) = Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^c \mathcal{O}_n(\mathbf{z}) \Theta_V(\tau_1, \tau_2, \epsilon),$$

for linear differential operator

$$\mathcal{O}_n(\mathbf{z}) = \sum_{\mathcal{G}^n} \mathcal{L}_\alpha (W(\mathcal{G}^n)), \quad (3.26)$$

where the sum is taken over all inequivalent order n Virasoro graphs \mathcal{G}^n .

Proof. We prove the result by induction in n . We have already shown the result holds for $n = 1$ and $n = 2$ and employ the Ward identity (3.7) to inductively prove (3.26) for $n \geq 2$.

Every inequivalent order n Virasoro graph \mathcal{G}^n can be characterized, according to the nature of the z_1 vertex, in terms of following five types:

- (i) $\deg(z_1) = 0$: $z_1 \circ \dots$
- (ii) $\deg(z_1) = 1$: $z_1 \circ \rightarrow \circ z_a \dots$ or $z_1 \circ \leftarrow \circ z_a \dots$
- (iii) $\deg(z_1) = 2$ where the z_1 -vertex forms a 1-cycle: $z_1 \circ \text{loop} \dots$
- (iv) $\deg(z_1) = 2$ where the z_1 -vertex is an element of a 2-cycle: $z_1 \circ \text{loop} \circ z_k \dots$
- (v) $\deg(z_1) = 2$ where either the z_1 -vertex is a non end-vertex of a chain or an element of an r -cycle with $r \geq 3$: $\dots z_a \circ \rightarrow \overset{z_1}{\circ} \rightarrow \circ z_b \dots$

The Ward identity (3.7) and (3.14) imply we may recursively describe $\mathcal{O}_n(\mathbf{z})$ as follows:

$$\begin{aligned}\mathcal{O}_n(\mathbf{z}) &= \frac{c}{12}s(z_1)\mathcal{O}_{n-1}(z_2, \dots, z_n) \\ &\quad + \left(\nabla_{z_1} + dz_1^2 \sum_{k=2}^n ({}^2\mathcal{P}_1(z_1, z_k)\partial_{z_k} + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k)) \right) \mathcal{O}_{n-1}(z_2, \dots, z_n) \\ &\quad + \frac{c}{2} \sum_{k=2}^n {}^2\mathcal{P}_4(z_1, z_k) dz_1^2 dz_k^2 \mathcal{O}_{n-2}(z_2, \dots, \widehat{z_k}, \dots, z_n),\end{aligned}\tag{3.27}$$

We now show how the parts of (3.27) relate to Virasoro graph weights by using induction in n . Thus given \mathcal{O}_{n-1} and \mathcal{O}_{n-2} satisfy (3.26), we see that the $\frac{c}{12}s(z_1)\mathcal{O}_{n-1}$ term of (3.27) arises from the sum over all \mathcal{G}^n graphs of type (iii).

Let \mathcal{G}^{n-1} denote an order $n-1$ Virasoro graph labelled by z_2, \dots, z_n of weight $W(\mathcal{G}^{n-1})$. This gives a contribution to (3.27) of

$$\nabla_{z_1} \mathcal{L}_\alpha (W(\mathcal{G}^{n-1})) = \mathcal{L}_\alpha (W(\mathcal{G}^{n-1})A(z_1, z_1)) + \mathcal{L}_\alpha (\nabla_{z_1} W(\mathcal{G}^{n-1})), \tag{3.28}$$

using the Leibniz rule for ∇_x . In particular, all terms of the form $W(\mathcal{G}^{n-1})A(z_1, z_1)$ arise as weights of \mathcal{G}^n graphs of type (i).

Let us examine the contributions that arise from $\nabla_{z_1} W(\mathcal{G}^{n-1})$ in (3.28) and the remaining terms in (3.27) and show that these can be expressed in terms of a sum of the weights of graphs of type (ii), (iv) and (v). Let z_k be a given vertex in \mathcal{G}^{n-1} for $k = 2, \dots, n$. Then, much as before, we can characterize \mathcal{G}^{n-1} according to (a) z_k is a degree 0 vertex (b) z_k is a disconnected vertex of degree 2 (c) z_k is a degree 1 vertex or (d) z_k is a degree 2 vertex in a chain or an r -cycle for $r \geq 2$.

Case (a). \mathcal{G}^{n-1} consists of a z_k vertex of degree 0 and an order $n-2$ Virasoro graph \mathcal{G}^{n-2} (with vertices $z_2, \dots, \widehat{z_k}, \dots, z_n$) of weight

$$W(\mathcal{G}^{n-1}) = A(z_k, z_k)W(\mathcal{G}^{n-2}).$$

Using (3.19) this contributes to (3.27) the term

$$\mathcal{D}_{z_1, z_k} A(z_k, z_k)W(\mathcal{G}^{n-2}) = 2A(z_1, z_k)\omega(z_1, z_k)W(\mathcal{G}^{n-2}),$$

the sum of the weights of two \mathcal{G}^n graphs of type (ii) where z_1 and z_k form a disconnected chain of length 2.

Case (b). \mathcal{G}^{n-1} consists of a disconnected degree 2 vertex z_k and an order $n-2$ Virasoro graph \mathcal{G}^{n-2} of weight $W(\mathcal{G}^{n-1}) = \frac{c}{12}s(z_k)W(\mathcal{G}^{n-2})$ which contributes $\frac{c}{12}\mathcal{D}_{z_1, z_k}(s(z_k))W(\mathcal{G}^{n-2})$ to (3.27). Summing with the $\frac{c}{2}{}^2\mathcal{P}_4(z_1, z_k)W(\mathcal{G}^{n-2})$ contribution to (3.27) gives

$$\frac{c}{2}\omega(z_1, z_k)^2 W(\mathcal{G}^{n-2}),$$

using (3.10), the weight of a \mathcal{G}^n graph of type (iv) where z_1 and z_k form a 2-cycle.

Case (c). z_k is an end-vertex of a chain $\mathcal{C}_{k\ell}$ so that $W(\mathcal{G}^{n-1}) = A(z_k, z_\ell)\omega(z_k, z_m)\dots$, where z_k is joined to z_m and the ellipsis denotes the factors independent of z_k . Using (3.8) and (3.9) this contributes terms to (3.27) of the form

$$\begin{aligned} & \left(\left(\nabla_{z_1} + dz_1^2 \left({}^2\mathcal{P}_1(z_1, z_k)\partial_{z_k} + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k) \right. \right. \right. \\ & \quad \left. \left. \left. + {}^2\mathcal{P}_1(z_1, z_m)\partial_{z_m} + {}^2\mathcal{P}_2(z_1, z_m) \right) \right) A(z_k, z_\ell)\omega(z_k, z_m) \right) \dots \\ & = A(z_1, z_\ell)\omega(z_1, z_k)\omega(z_k, z_m)\dots + A(z_k, z_\ell)\omega(z_k, z_1)\omega(z_1, z_m)\dots \end{aligned} \quad (3.29)$$

Note that we have omitted in (3.29) contributions to (3.27) of the form:

$$A(z_k, z_\ell)\omega(z_k, z_m) \left(\nabla_{z_1} + {}^2\mathcal{P}_1(z_1, z_m)\partial_{z_m} + {}^2\mathcal{P}_2(z_1, z_m) \right) (\dots)$$

which contribute to case (d) for z_m . The first term in (3.29) is the weight of a \mathcal{G}^n graph of type (ii):

$$z_1 \circ \longrightarrow \overset{z_k}{\circ} \longrightarrow \overset{z_m}{\circ} \quad \dots \quad \longrightarrow \circ z_\ell \quad \dots$$

and the second term is the weight of a graph of type (v):

$$z_k \circ \longrightarrow \overset{z_1}{\circ} \longrightarrow \overset{z_m}{\circ} \quad \dots \quad \longrightarrow \circ z_\ell \quad \dots$$

Case (d). If $\deg(z_k) = 2$ then $W(\mathcal{G}^{n-1}) = \omega(z_a, z_k)\omega(z_k, z_b)\dots$, where z_k is joined to z_a and z_b and the ellipsis denotes the factors independent of z_k . This contributes terms to (3.27) of the form

$$\begin{aligned} & \left(\left(\nabla_{z_1} + dz_1^2 \left({}^2\mathcal{P}_1(z_1, z_k)\partial_{z_k} + {}^2\mathcal{P}_1(z_1, z_a)\partial_{z_a} + {}^2\mathcal{P}_1(z_1, z_b)\partial_{z_b} \right. \right. \right. \\ & \quad \left. \left. \left. + {}^2\mathcal{P}_2(z_1, z_a) + 2 \cdot {}^2\mathcal{P}_2(z_1, z_k) + {}^2\mathcal{P}_2(z_1, z_b) \right) \right) \omega(z_a, z_k)\omega(z_k, z_b) \right) \dots \\ & = \omega(z_a, z_1)\omega(z_1, z_k)\omega(z_k, z_b)\dots + \omega(z_a, z_k)\omega(z_k, z_1)\omega(z_1, z_b)\dots \end{aligned} \quad (3.30)$$

using (3.8). Note that we have omitted in (3.8) contributions to (3.27) of the form:

$$\begin{aligned} & \omega(z_a, z_k)\omega(z_k, z_a) \left(\nabla_x + {}^2\mathcal{P}_1(z_1, z_a)\partial_{z_a} + {}^2\mathcal{P}_1(z_1, z_b)\partial_{z_b} \right. \\ & \quad \left. + {}^2\mathcal{P}_2(z_1, z_a) + {}^2\mathcal{P}_2(z_1, z_b) \right) (\dots) \end{aligned}$$

which contribute to case (c) and case (d) for z_a or z_b . The two terms in (3.30) are weights of a \mathcal{G}^n graphs of type (v):

$$\dots z_a \circ \longrightarrow \overset{z_1}{\circ} \longrightarrow \overset{z_k}{\circ} \longrightarrow \circ z_b \dots, \quad \dots z_a \circ \longrightarrow \overset{z_k}{\circ} \longrightarrow \overset{z_1}{\circ} \longrightarrow \circ z_b \dots$$

Thus, altogether, we find that the weights of all \mathcal{G}^n graphs of type (i)-(v) contribute and hence (3.26) holds. \square

Remark 3.12. $\mathcal{O}_n(\mathbf{z})$ of (3.26) is symmetric in z_1, \dots, z_n and is expressed in a coordinate free way in terms of $\omega(z_i, z_j)$, $\nu_a(z_i)$, $s(z_i)$ and $\frac{\partial}{\partial \Omega_{ab}}$ for all $\Omega \in \mathbb{H}_2$ where the **only** dependence on the original VOA is the central charge c . Furthermore, using Corollaries 3.5 and 3.6 it follows that $\mathcal{O}_n(\mathbf{z})$ satisfies the recurrence relation:

$$\begin{aligned} \mathcal{O}_n(\mathbf{z}) = & \left(\nabla_{z_1} + \frac{c}{12} s(z_1) + \sum_{k=2}^n dz_k (\Psi(z_1, z_k) \partial_{z_k} + 2 \partial_{z_k} \Psi(z_1, z_k)) \right) \mathcal{O}_{n-1}(z_2, \dots, z_n) \\ & + 3c \sum_{k=2}^n dz_k^3 (\partial_{z_k}^3 \Psi(z_1, z_k)) \mathcal{O}_{n-2}(z_2, \dots, \widehat{z}_k, \dots, z_n), \end{aligned} \quad (3.31)$$

for any choice of coordinates \mathbf{z} and for all $\Omega \in \mathbb{H}_2$.

Remark 3.13. Theorem 3.11 can be readily generalized for any pair of ordinary V -modules W_1, W_2 with genus two n -point function

$$\begin{aligned} Z_{W_1, W_2}^{(2)}(\tilde{\omega}, z_1; \dots; \tilde{\omega}, z_n) d\mathbf{z}^2 &= d\mathbf{z}^2 \sum_{u \in V} Z_{W_1}^{(1)}(Y[\tilde{\omega}, z_1] \dots Y[\tilde{\omega}, z_n] u; \tau_1) Z_{W_2}^{(1)}(\bar{u}; \tau_2) \\ &= \mathcal{O}_n(\mathbf{z}) \Theta_{W_1, W_2}(\tau_1, \tau_2, \epsilon), \end{aligned}$$

where $\Theta_{W_1, W_2}(\tau_1, \tau_2, \epsilon) = Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^{-c} Z_{W_1, W_2}^{(2)}(\tau_1, \tau_2, \epsilon)$. Following Conjecture 3.9 we further conjecture that if V is rational and C_2 cofinite, then Θ_{W_1, W_2} for irreducible W_1, W_2 form further components of a weight $c/2$ vector valued Siegel modular form.

4 Analytic and modular transformations

By remark 3.12, we may express $\mathcal{O}_n(\mathbf{z})$ in any coordinate system on an arbitrary genus two Riemann surface. In particular, we can consider the behaviour of $\mathcal{O}_n(\mathbf{z})$ under a general analytic transformation:

Proposition 4.1. Let $z \rightarrow \phi(z)$ be an analytic map then we have

$$\mathcal{O}_n(\mathbf{z}) = \mathcal{O}_n(z_1, \dots, \phi(z_i), \dots, z_n) + \frac{c}{12} \{ \phi(z_i), z_i \} dz_i^2 \mathcal{O}_{n-1}(z_1, \dots, \widehat{z}_i, \dots, z_n),$$

for $i \in \{1, \dots, n\}$ and $\{ \phi(z), z \}$ is the Schwarzian derivative.

Proof. Choose $i = 1$ wlog by Proposition 3.1. $\omega(z_1, z_j), \nu_a(z_1)$ are invariant under an analytic transformation whereas $s(z_1)$ transforms as in (2.2). Such a $s(z_1)$ term only arises in the Virasoro graphs \mathcal{G}^n of type (iii), where the z_1 -vertex forms a 1-cycle of weight $W(\mathcal{G}^n) = \frac{c}{12} s(z_1) W(\mathcal{G}^{n-1})$. Thus the result follows. \square

In order to describe the genus two modular properties of $\mathcal{O}_n(\mathbf{z})$ we analyse the modular properties of the $(2, -1)$ -form $\Psi(x, y)$ of (3.5). The genus two modular group $\mathrm{Sp}(4, \mathbb{Z})$ consists of integral block matrices $\gamma := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D obey:

$$\begin{aligned} A^T D - C^T B &= I, & AB^T &= BA^T, & CD^T &= DC^T, \\ A^T C &= C^T A, & B^T D &= D^T B, \end{aligned} \quad (4.1)$$

for identity matrix I . It is convenient to define for $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ and $\Omega \in \mathbb{H}_2$

$$M = C\Omega + D, \quad N = (C\Omega + D)^{-1}.$$

The holomorphic differentials $\boldsymbol{\nu}(x)$, the period matrix Ω , the derivative operator ∇_x , the bidifferential $\omega(x, y)$ and the projective connection $s(x)$ transform under $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ as follows [F, Mu1, GT1]

$$\boldsymbol{\nu}^\gamma(x) = \boldsymbol{\nu}N, \quad \Omega^\gamma = (A\Omega + B)N, \quad \nabla_x^\gamma = \nabla_x, \quad (4.2)$$

$$\omega^\gamma(x, y) = \omega(x, y) - \frac{1}{2} \sum_{1 \leq a \leq b \leq 2} (\nu_a(x)\nu_b(y) + \nu_b(x)\nu_a(y)) \frac{\partial}{\partial \Omega_{ab}} \log \det M, \quad (4.3)$$

$$s^\gamma(x) = s(x) - 6 \nabla_x \log \det M. \quad (4.4)$$

We now show that for the the $(2, -1)$ bidifferential $\Psi(x, y)$ of (3.5)

Theorem 4.2. $\Psi(x, y)$ is $\mathrm{Sp}(4, \mathbb{Z})$ modular invariant.

In order to prove Theorem 4.2 we need two lemmas. The first lemma concerns the second term on the right hand side of (4.3):

Lemma 4.3.

$$\frac{1}{2} \sum_{1 \leq a \leq b \leq 2} (\nu_a(x)\nu_b(y) + \nu_b(x)\nu_a(y)) \frac{\partial}{\partial \Omega_{ab}} \log \det M = \boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y). \quad (4.5)$$

Proof. Using the $\mathrm{Sp}(4, \mathbb{Z})$ relations (4.1) we find $N = A^T - C^T\Omega^\gamma$ so that

$$(NC)^T = NC. \quad (4.6)$$

The result follows by direct calculation using (4.6) where we find

$$\begin{aligned} \partial_{11} \log \det M &= (M_{22}C_{11} - M_{12}C_{21})/\det M = (NC)_{11}, \\ \partial_{22} \log \det M &= (NC)_{22}, \\ \partial_{12} \log \det M &= 2(NC)_{12}. \end{aligned}$$

□

Lemma 4.4. For $\boldsymbol{\nu}^\gamma(x) = \boldsymbol{\nu}N$ of (4.2) we have

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(x) \\ \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N, \quad (4.7)$$

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(y) \\ \partial_y \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(y) \\ \partial_y \boldsymbol{\nu}(y) \end{vmatrix} \det N, \quad (4.8)$$

$$\begin{vmatrix} \boldsymbol{\nu}^\gamma(y) \\ \nabla_x^\gamma \boldsymbol{\nu}^\gamma(y) \end{vmatrix} = \begin{vmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{vmatrix} \det N + \boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y) \begin{vmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{vmatrix} \det N. \quad (4.9)$$

Proof. $\left| \begin{smallmatrix} \boldsymbol{\nu}^\gamma(x) \\ \boldsymbol{\nu}^\gamma(y) \end{smallmatrix} \right| = \left| \begin{smallmatrix} \boldsymbol{\nu}(x)N \\ \boldsymbol{\nu}(y)N \end{smallmatrix} \right| = \left| \begin{smallmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{smallmatrix} \right| \det N$ and similarly for (4.8). To prove (4.9), we first note that

$$\nabla_x M = C\boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x).$$

Furthermore, $(\nabla_x N)M = -N\nabla_x M$ so that

$$\nabla_x N = -N(\nabla_x M)N = -NC\boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x)N. \quad (4.10)$$

Hence, using (4.2) and (4.6), we find that

$$\begin{aligned} \left| \begin{smallmatrix} \boldsymbol{\nu}^\gamma(y) \\ \nabla_x^\gamma \boldsymbol{\nu}^\gamma(y) \end{smallmatrix} \right| &= \left| \begin{smallmatrix} \boldsymbol{\nu}(y)N \\ \nabla_x(\boldsymbol{\nu}(y))N + \boldsymbol{\nu}(y)\nabla_x N \end{smallmatrix} \right| \\ &= \left| \begin{smallmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{smallmatrix} \right| \det N + \left| \begin{smallmatrix} \boldsymbol{\nu}(y)N \\ -\boldsymbol{\nu}(y)NC\boldsymbol{\nu}(x)^T \boldsymbol{\nu}(x)N \end{smallmatrix} \right| \\ &= \left| \begin{smallmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{smallmatrix} \right| \det N + \boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y) \left| \begin{smallmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{smallmatrix} \right| \det N. \end{aligned}$$

□

Proof of Theorem 4.2. Combining Lemmas 4.3 and 4.4 we immediately obtain

$$\omega^\gamma(x, y) \left| \begin{smallmatrix} \boldsymbol{\nu}^\gamma(x) \\ \boldsymbol{\nu}^\gamma(y) \end{smallmatrix} \right| + \left| \begin{smallmatrix} \boldsymbol{\nu}^\gamma(y) \\ \nabla_x^\gamma \boldsymbol{\nu}^\gamma(y) \end{smallmatrix} \right| = \left[\omega(x, y) \left| \begin{smallmatrix} \boldsymbol{\nu}(x) \\ \boldsymbol{\nu}(y) \end{smallmatrix} \right| + \left| \begin{smallmatrix} \boldsymbol{\nu}(y) \\ \nabla_x \boldsymbol{\nu}(y) \end{smallmatrix} \right| \right] \det N.$$

Thus using (4.8) we find $\Psi(x, y)$ is $\text{Sp}(4, \mathbb{Z})$ modular invariant. □

Corollary 4.5. *The differential equations (3.11)-(3.13) are $\text{Sp}(4, \mathbb{Z})$ invariant.*

Proof. Under the action of $\gamma \in \text{Sp}(4, \mathbb{Z})$, the change in the left hand side of (3.11) is

$$\begin{aligned} & -\nabla_x (\boldsymbol{\nu}(y_1)NC\boldsymbol{\nu}^T(y_2)) - \sum_{r=1}^2 \partial_{y_r} (\Psi(x, y_r)\boldsymbol{\nu}(y_1)NC\boldsymbol{\nu}^T(y_2)) dy_r \\ &= -\sum_{r=1}^2 \omega(x, y_r)\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_r) - \boldsymbol{\nu}(y_1)(\nabla_x N)C\boldsymbol{\nu}^T(y_2), \end{aligned}$$

using (3.12). But (4.6) and (4.10) imply

$$-\boldsymbol{\nu}(y_1)(\nabla_x N)C\boldsymbol{\nu}^T(y_2) = (\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_1)) (\boldsymbol{\nu}(x)NC\boldsymbol{\nu}^T(y_2)).$$

Thus the total change in the left hand side of (3.11) is

$$\omega^\gamma(x, y_1)\omega^\gamma(x, y_2) - \omega(x, y_1)\omega(x, y_2),$$

as required. A similar method shows that (3.12) is modular invariant. Lastly, $\partial_y^k \Psi(x, y)$ is modular invariant so that modular invariance of (3.13) follows by considering the $y_1 \rightarrow y_2$ limit in the above analysis as described in the proof of Corollary 3.6. □

Let us now consider the modular properties of the operator $\mathcal{O}_n(\mathbf{z})$. Following Remark 3.12 we know that $\mathcal{O}_n(\mathbf{z})$ depends on $\omega(x, y)$, $\nu_1(x)$, $\nu_2(x)$, $s(x)$ and $\frac{\partial}{\partial \Omega_{ab}}$ for all $\Omega \in \mathbb{H}_2$. These terms transform under $\gamma \in \text{Sp}(4, \mathbb{Z})$ as in (4.2)-(4.4) so that

$$\mathcal{O}_n(\mathbf{z}) \rightarrow \mathcal{O}_n^\gamma(\mathbf{z}).$$

We then find

Theorem 4.6. *For differentiable $F = F(\Omega)$ we have for all $\gamma \in \text{Sp}(4, \mathbb{Z})$ that*

$$\mathcal{O}_n^\gamma(\mathbf{z}) (\det(M)^{c/2} F) = \det(M)^{c/2} \mathcal{O}_n(\mathbf{z}) F. \quad (4.11)$$

Proof. We prove the result by induction in n . The result is trivially true for $n = 0$. For $n = 1$ we use (3.17) to find

$$\begin{aligned} \mathcal{O}_1^\gamma(z_1) (\det(M)^{c/2} F) &= \left(\nabla_{z_1}^\gamma + \frac{c}{12} s^\gamma(z_1) \right) (\det(M)^{c/2} F) \\ &= \det(M)^{c/2} \mathcal{O}_1(z_1) F, \end{aligned}$$

using (4.2) and (4.4). (3.31) implies by induction that for $n \geq 2$

$$\begin{aligned} &\mathcal{O}_n^\gamma(\mathbf{z}) (\det(M)^{c/2} F) \\ &= \left(\nabla_{z_1}^\gamma + \frac{c}{12} s^\gamma(z_1) + \sum_{k=2}^n dz_k (\Psi(z_1, z_k) \partial_{z_k} + 2 \partial_{z_k} \Psi(z_1, z_k)) \right) (\det(M)^{c/2} \mathcal{O}_{n-1} F) \\ &\quad + 3c \det(M)^{c/2} \sum_{k=2}^n dz_k^3 (\partial_{z_k}^3 \Psi(z_1, z_k)) \mathcal{O}_{n-2} F \\ &= \det(M)^{c/2} \mathcal{O}_n(\mathbf{z}) F, \end{aligned}$$

using (4.2) and (4.4) again. Thus the result follows. \square

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